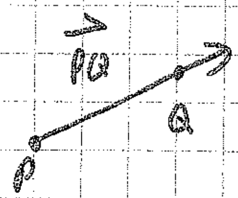
Unit 7 – Vectors

7.1 – Algebraic Vectors

<u>Objectives</u> – You will ① find ordered pairs that represent vectors
② Add, subtract, multiply, and find the magnitude of vectors algebraically

<u>Vector Designation</u>: $\vec{v}$ – all vectors have <u>magnitude</u> and <u>direction</u>.

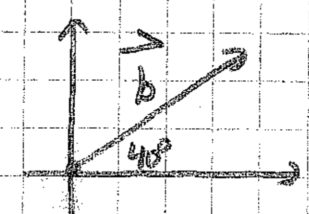


<u>magnitude</u> – length of a line segment.

<u>direction</u> – indicated by the arrow.

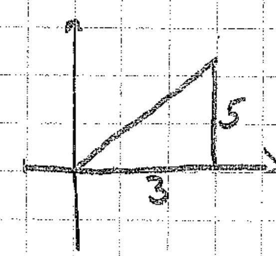
<u>Standard Position</u>: when a vector starts at the origin.

<u>Direction</u>: the angle between the positive x-axis and the vector.



<u>Example</u>:

Find the magnitude of $\langle 3, 5 \rangle$



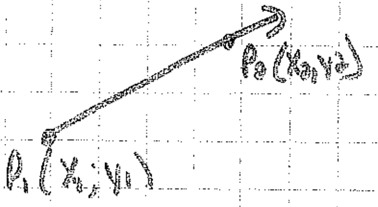
$3^2 + 5^2 = x^2$

$9 + 25 = x^2$

$\sqrt{34} = \sqrt{x^2}$

$\boxed{x = \sqrt{34}}$

Representing a Vector as an Ordered Pair



$\overrightarrow{P_1P_2} = \langle x_2 - x_1 , y_2 - y_1 \rangle$

$\text{Magnitude} = |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Ex: Find the ordered pair that represents the vector from $A(7-1)$ to $B(-2,-5)$ and find the magnitude.

$$\overrightarrow{AB} = \langle x_2 - x_1,\ y_2 - y_1 \rangle$$
$$= \langle -2-7,\ -5-(-1) \rangle$$
$$= \langle -9, -4 \rangle$$

$$|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$
$$= \sqrt{(-2-7)^2 + (-5-(-1))^2}$$
$$= \sqrt{(-9)^2 + (-4)^2}$$
$$= \sqrt{81 + 16} = \boxed{\sqrt{97}}$$

Unit Vectors

A unit vector has a magnitude of 1.

A unit vector in the direction of the x-axis is represented by i and in the direction of the y-axis is represented by $\vec{j}$

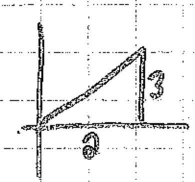
ex: Write $\langle 3, 5 \rangle$ as the sum of unit vectors.

$$\langle 3, 5 \rangle = 3\vec{i} + 5\vec{j}$$

ex: Write $\langle 2, 3 \rangle$ as the sum of unit vectors and find the magnitude.

$$\langle 2, 3 \rangle = \boxed{2\vec{i} + 3\vec{j}}$$

Magnitude



$$2^2 + 3^2 = x^2$$
$$13 = x^2$$
$$\boxed{x = \sqrt{13}}$$

Magnitude = 1

Vector Operations

ex: Given $\vec{b} = \langle 6, 3 \rangle$ and $\vec{c} = \langle -4, 8 \rangle$

Evaluate: $\vec{a}$ if $\vec{a} = 6\vec{b} - 4\vec{c}$

$$\vec{a} = 6\langle 6, 3 \rangle - 4\langle -4, 8 \rangle$$
$$= \langle 36, 18 \rangle - \langle -16, 32 \rangle$$
$$= \langle 52, -14 \rangle$$

Homework:

1) Given $A\langle 6,3\rangle$ and $B\langle -4,5)$

a) Find ordered pair $\overrightarrow{AB}$

b) Magnitude of $\overrightarrow{AB}$

c) Write as the sum of unit vectors

2) If $\vec{b} = \langle -3,1\rangle$ and $\vec{c} = \langle 5,6\rangle$ find

a) $\vec{d} = 2\vec{b} + \vec{c}$

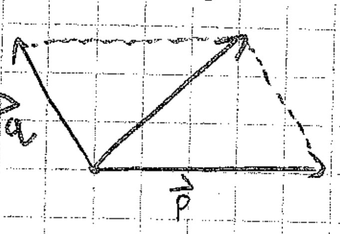
b) $\vec{e} = 3\vec{b} - 2\vec{c}$

7-2 Geometric Vectors

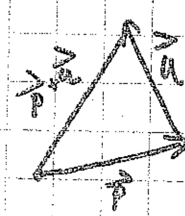
Objective – You will (1) Find equal, opposite and parallel vectors
(2) Add and subtract vectors geometrically.

Geometric Vectors - when 2 or more forces act on an object, the resulting force can be found using one of two methods:

Parallelogram Method

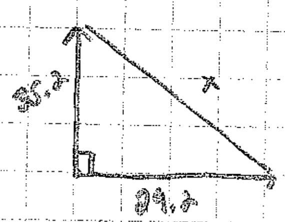


Triangle Method (Tip-to-Tail)



Examples:

1) The magnitude of m is 29.2 and the magnitude of n is 35.2. If $m \perp n$, what is the magnitude of the resultant?



$(35.2)^2 + (29.2)^2 = x^2$

$1239.04 + 852.64 = x^2$

$\sqrt{2091.68} = \sqrt{x^2}$

$x = 45.73488821$

$\boxed{x = 45.73}$

2) An airplane is flying west at a velocity of 100 m/sec. If the wind is blowing out of the south at 5 m/sec, find the magnitude of the resultant of the plane's path.

$5^2 + 100^2 = x^2$

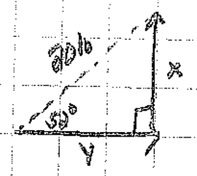
$25 + 10,000 = x^2$

$\sqrt{10,025} = \sqrt{x^2}$

$x = 100.12492$

$\boxed{x = 100.12 \text{ m/sec}}$

3) If Molly is pulling a toy by exerting a force of 20 lbs on a string attached to a toy. The string makes an angle of 52° with the floor. Find the vertical and horizontal components of the force.



$\sin 52 = \dfrac{x}{20}$

$x = 20 \cdot \sin 52$

$x = 15.76021507$

$\boxed{x = 15.76 \text{ lbs}}$

$\cos 52 = \dfrac{y}{20}$

$y = 20 \cdot \cos 52$

$y = 12.31322951$

$\boxed{y = 12.31 \text{ lbs}}$

Homework:

1) Two forces are perpendicular and their magnitudes are 1180 lbs and 27.30 lbs. Find the magnitude of their resultant.

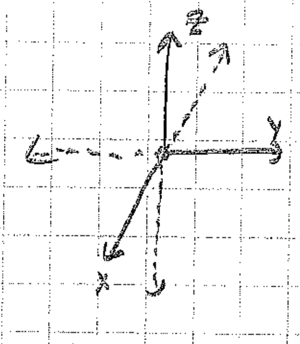
and vertical components

2) A car is traveling at a speed of 80 mph at an angle of 32° north of east. Find the vertical and horizontal components.

7-3 Perpendicular Vectors

Objective – You will ① Find the inner and cross product of two vectors.
② Determine whether two vectors are $\perp$.

Vectors in 3D



ex: Given: $X: (5, 3, 2)$ & $Y(4, -5, 6)$

a) Write xy as an ordered triple.

$$\overrightarrow{xy} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$
$$= (4-5,\ -5-(-3),\ 6-2)$$

$$\overrightarrow{xy} = \langle -1, -2, 4 \rangle$$

b) Write xy as the sum of unit vectors.

$$= -1\vec{i} - 2\vec{j} + 4\vec{k}$$

c) Find the magnitude of xy.

$$|\overrightarrow{xy}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$
$$= \sqrt{(-1)^2 + (-2)^2 + (4)^2} = \sqrt{1+4+16}$$
$$\boxed{= \sqrt{21}}$$

Perpendicular Vectors

Inner Product (Dot Product)

- used to determine if vectors are perpendicular.
- if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$
- if inner product is 0, the vectors are $\perp$.

Example

Determine which vectors are $\perp$ if $\vec{a} = \langle 4, 2 \rangle$, $\vec{b} = \langle -3, 6 \rangle$ and $\vec{c} = \langle 2, 1 \rangle$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 4(-3) + 2(6) \\ &= -12 + 12 \\ &= 0 \end{aligned}$$

$\therefore \vec{a} \perp \vec{b}$

$$\begin{aligned} \vec{a} \cdot \vec{c} &= 4(2) + 2(1) \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

$\therefore$ not $\perp$

$$\begin{aligned} \vec{b} \cdot \vec{c} &= -3(2) + 6(1) \\ &= -6 + 6 \\ &= 0 \end{aligned}$$

$\therefore \vec{b} \perp \vec{c}$

Cross Product

- If a vector has a third-dimension, you can use matrix properties to find the vector that is $\perp$ to two given vectors.

- You can use inner product to verify that your cross product is $\perp$ to the original vectors.

Example

Given $\vec{d} = \langle 1, 3, 2 \rangle$ and $\vec{e} = \langle 2, -1, -1 \rangle$

a) Find the cross product.

$$\begin{vmatrix} i & j & k & i & j \\ 1 & 3 & 2 & 1 & 3 \\ 2 & -1 & -1 & 2 & -1 \end{vmatrix} = i(3)(-1) + j(2)(2) + k(1)(-1) - j(1)(-1) - i(2)(-1) - k(3)(2)$$

$$= -3\vec{i} + 4\vec{j} + -1\vec{k} + (+1)\vec{j} + (+2)\vec{i} - 6\vec{k}$$

$$= -1\vec{i} + 5\vec{j} - 7\vec{k}$$

b) Verify the cross product is $\perp$ to the original vectors.

$-1\vec{i} + 5\vec{j} - 7\vec{k}$
$= \langle -1, 5, -7 \rangle$

$\langle -1, 5, -7 \rangle \cdot \langle 1, 3, 2 \rangle$
$= -1(1) + 5(3) - 7(2)$

$\langle -1, 5, -7 \rangle \cdot \langle 2, -1, -1 \rangle$
$= -1(2) + 5(-1) + -7(-1)$

Homework:

1) Determine if the two vectors are $\perp$.

$\vec{a} = \langle 3, 5 \rangle \quad \vec{b} = \langle 4, -2 \rangle$

2) Determine if the two vectors are $\perp$.

$\vec{r} = \langle -2, 4, 8 \rangle \quad \vec{s} = \langle 16, 4, 2 \rangle$

3) Find the cross product and then use the inner product to show that it is $\perp$ to the 2 original vectors.

$\vec{x} = \langle 5, 2, 3 \rangle$

$\vec{y} = \langle -2, 5, 0 \rangle$

7-4 – Applications of Vectors and Multiple Forces.

<u>Objective</u> – You will solve problems using vectors and right △ Trig.

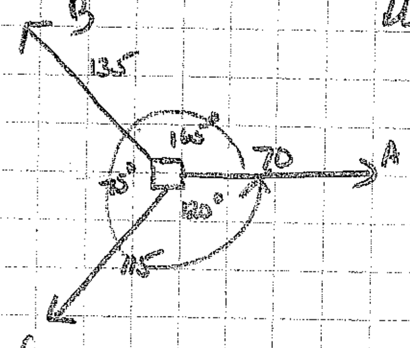
<u>Multiple Forces</u>

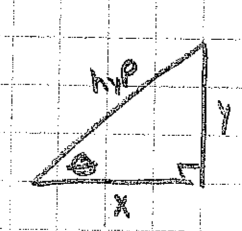
1) Find the horizontal and vertical components of each given force.

2) Find the sum of the horizontal and vertical components.

3) Using the sums, find the resultant and direction (must include angle and N, S, E, W).

<u>Example #1</u> Find the magnitude and direction of the resultant force.





$\cos\theta = \frac{x}{hyp}$

$\boxed{x = hyp \cdot \cos\theta}$

$\sin\theta = \frac{y}{hyp}$

$\boxed{y = hyp \cdot \sin\theta}$

$A_x: 70 \cdot \cos 0 = \boxed{70}$

$A_y: 70 \cdot \sin\theta = \boxed{0}$

$B_x: 135 \cdot \cos 165 = -130.3999865$

$\boxed{= -130.4}$

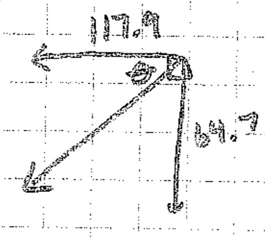
$B_y: 135 \cdot \sin 165 = 34.94057$

$\boxed{= 34.9}$

$C_x: 115 \cdot \cos 240 = \boxed{-57.5}$

$C_y: 115 \cdot \sin 240 = -99.59292$

$= \boxed{-99.6}$



$\text{Magnitude} = \sqrt{x^2 + y^2}$

$= \sqrt{(117.9)^2 + (64.7)^2}$

$\text{Mag} = \boxed{134.486}$

$\sum F_x = 70 - 130.4 - 57.5 = -117.9$

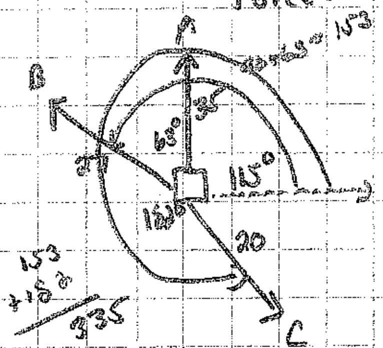
$\sum F_y = 0 + 34.9 - 99.6 = -64.7$

$\theta = \tan^{-1}\left(\frac{64.7}{117.9}\right) = 28.8^\circ$

$\boxed{28.8^\circ \text{ S of W}}$

Example 2

Find the magnitude and direction of the resultant force:



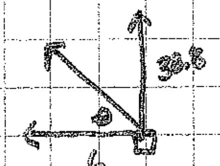
$A_x = 35 \cdot \cos 90 = 0$
$A_y = 35 \cdot \sin 90 = 35$

$B_x = 27 \cdot \cos 153 = -24.1$
$B_y = 27 \cdot \sin 153 = 12.3$

$C_x = 20 \cos 335 = 18.1$
$C_y = 20 \sin 335 = -8.5$

$\sum F_x = 0 + (-24.1) + 18.1 = \boxed{-6}$

$\sum F_y = 35 + 12.3 + (-8.5) = \boxed{38.8}$

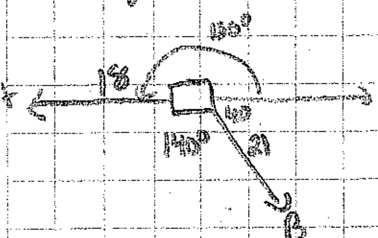


$\text{Magnitude} = \sqrt{(38.8)^2 + (6)^2}$
$= 39.26117675$
$\boxed{= 39.26}$

$\theta = \tan^{-1}\left(\frac{38.8}{6}\right)$
$\boxed{= 81.2°}$

$\boxed{81.2° \text{ N of W}}$

Example #3 Based on the diagram provided, find the magnitude and direction of a third force that produces equilibrium.

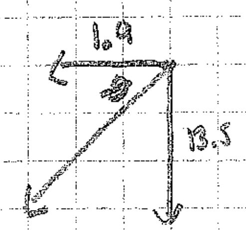


$A_x: 18 \cdot \cos 180 = -18$
$A_y: 18 \cdot \sin 180 = 0$

$B_x: 21 \cdot \cos 320 = \boxed{16.1}$
$B_y: 21 \cdot \sin 320 = \boxed{-13.5}$

$\sum F_x = -18 + 16.1 = \boxed{-1.9}$

$\sum F_y = 0 - 13.5 = \boxed{-13.5}$



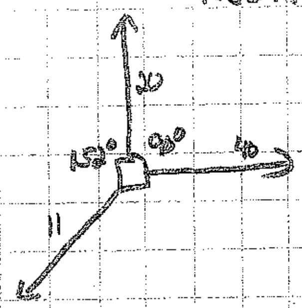
$\text{Magnitude} = \sqrt{(1.9)^2 + (13.5)^2}$
$\boxed{= 13.63}$

$\theta = \tan^{-1}(13.5/1.9) = 81.988$
$= 81.99°$ S of W

$\boxed{\therefore \text{ Equilibrium @ } 81.99° \text{ N of E}}$

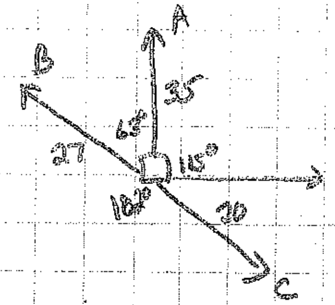
Homework

Find the magnitude and direction of the resultant force:



7-5 Multiple Forces (Day #2)

1) Find the magnitude and direction of the resultant force.

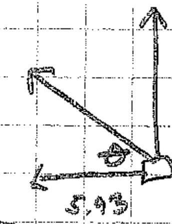


$A_x: 35 \cdot \cos 90 = 0$
$A_y: 35 \cdot \sin 90 = 35$

$B_x: 27 \cdot \cos 153 = -24.05717615 = -24.06$
$B_y: 27 \cdot \sin 153 = 12.25774349 = 12.26$

$C_x: 20 \cdot \cos 335 = 18.12615574 = 18.13$
$C_y: 20 \cdot \sin 335 = -8.452365235 = -8.45$

$\sum F_x: 0 + (-24.06) + 18.13 = -5.93$



$\sum F_y: 35 + 12.26 - 8.45 = 38.81$

Magnitude $= \sqrt{(5.93)^2 + (38.81)^2}$
$= 39.26042537 = \boxed{39.26}$

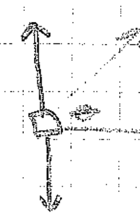
$\theta = \tan^{-1}(38.81/5.93) = 81.31264354 = \boxed{81.31° \text{ N of W}}$

Example #2: A 23 lb force acting at 60° above the horizontal and a 23 lb force acting at 120° above the horizontal act concurrently on a point. What is the magnitude and direction of a third force that produces equilibrium?

$A_x: 23 \cdot \cos 60 = 11.5$
$A_y: 23 \cdot \sin 60 = 19.91858429 = 19.92$

$B_x: 23 \cos 120 = -11.5$
$B_y: 23 \sin 120 = 19.9185$

$\sum F_x: 11.5 + (-11.5) = 0$



Mag = 39.84
$\theta = 0°$ S

$\sum F_y: 19.92 + 19.92 = 39.84$

7-6 Parametric Equations

Objective – You will: Write parametric equations of lines.

Parametric Equations

Equations that are written in terms of time.

$y = mx + b$ in parametric form would be:

$x = t$ — x is independent

$y = mt + b$ — y is dependent.

t is independent.

Examples: Write the following equations in parametric form

1) $y = 5x - 3$

$x = t$

$\boxed{y = 5t - 3}$

2) $2x - 5y = -3$

$\frac{-5y}{-5} = \frac{-2x - 3}{-5}$

$y = \frac{2}{5}x + \frac{3}{5}$

$x = t$

$\boxed{y = \frac{2}{5}t + \frac{3}{5}}$

Examples: Write the following equations in slope-intercept form:

3) $x = 4t - 2 \rightarrow \frac{x+2}{4} = \frac{4t}{4}$

$\boxed{t = \frac{x+2}{4}}$

$y = -2t + 4$

$\frac{y-4}{-2} = \frac{-2t}{-2}$

$\boxed{t = \frac{-y+4}{2}}$

$\frac{x+2}{4} = \frac{-y+4}{2}$

$2(x+2) = 4(-y+4)$

$2x + 4 = -4y + 16$

$\frac{2x - 12}{-4} = \frac{-4y}{-4}$

$\boxed{y = -\frac{1}{2}x + 3}$

or sub $\frac{x+2}{4}$ into 2nd equation

4) $3x = -5t - 4 \longrightarrow \frac{3x+4}{-5} = \frac{-5t}{-5}$

$2y = -2t + 3$

$t = \frac{-3x-4}{5}$

$\frac{2y-3}{-2} = \frac{-2t}{-2}$

$t = \frac{-2y+3}{2}$

$\frac{-3x-4}{5} = \frac{-2y+3}{2}$

$5(-2y+3) = 2(-3x-4)$

$-10y + 15 = -6x - 8$

$\quad -15 \qquad\quad -15$

$\frac{-10y}{-10} = \frac{-6x - 23}{-10}$

$\boxed{y = \frac{3}{5}x + \frac{23}{10}}$

Homework:

1) Write $-3y = 2x + 6$ in parametric form

2) Write $4y - 2x = 11$ in parametric form

3) Solve the following pair of parametric equations and write your answer in slope-intercept form.

$2x = 6t + 10$
$-3y + 6t = -7$

7-7 Modeling Motion Using Parametric Equations

<u>Objective</u>: You will: ① Use parametric equations to model motion

② Solve problems related to the motion, trajectory + Range of a projectile.

<u>Projectile</u> - an object that is launched.

<u>Trajectory</u> - path of the projectile

<u>Range</u> - the horizontal distance the projectile travels.

Horizontal and Vertical Components

<u>Initial Velocity</u>

<u>Horizontal</u>:

$x = v \cdot \cos\theta$

<u>Vertical</u>:

$y = v \cdot \sin\theta$

<u>After t Seconds</u>

<u>Horizontal</u>:

Distance = Rate x Time

$x = t \cdot v \cdot \cos\theta$

<u>Vertical</u>:

Distance = Rate x Time - Gravity + Initial Ht.

$y = t \cdot v \cdot \sin\theta - 16t^2 + h_0$

Mike Trout comes to bat with runners on first and third. Clayton Kershaw throws a slider across the plate about waist high, 3 feet above the ground. Jeter hits the ball with an initial velocity of 155 ft/s at an angle of 22° above the horizontal. The ball travels straight at the 415 foot mark on the center field wall which is 15 feet high.

a. Write parametric equations that describe the path of the ball.

b. Find the height of the ball after it has traveled 415 feet horizontally. Will the ball clear the fence or will the center fielder be able to catch it?

c. If there were no outfield seats, how far would the ball travel before it hits the ground?

a) $X = t \cdot v \cdot \cos\theta$ $\quad$ $y = t \cdot v \cdot \sin\theta - 16t^2 + h_0$

$\boxed{X = 155t \cos 22}$ $\quad$ $\boxed{y = 155t \sin 22 - 16t^2 + 3}$

b) $\dfrac{415}{155\cos 22} = \dfrac{155t\cos 22}{155\cos 22}$

$t = 2.88768$

$\boxed{t = 2.89 \text{ secs}}$

Fence is 15 ft high.

$y = 155(2.89)\sin 22 - 16(2.89)^2 + 3$

$y = 37.2508461$

$y = 37.25$ ft

∴ Yes it will clear the fence!!

c) Lands when $y = 0$

$0 = 155t \sin 22 - 16t^2 + 3$

on calc ⇒ #5 intersect

$t = 3.6799531$

$X = 155(3.6799531)\cos 22$

$X = 528.8589305$

∴ The ball would hit the ground at 528.86 ft!!

Jordan Spieth hits a golf ball with an initial velocity of 150 feet per second at an angle of $25°$ above the horizontal. He estimates the distance to the hole to be 200 yards.

a. Write the position of the ball as a pair of parametric equations.

b. Find the range of the ball.

c. If the green is 20 yards in diameter with the hole in the center, would the ball land in the green?

a) $X = t \cdot v \cdot \cos\theta$ $\qquad y = t \cdot v \cdot \sin\theta - 16t^2 + h_0$

$X = 150t\cos 25$ $\qquad y = 150t\sin 25 - 16t^2 + 0$

$y = 150t\sin 25 - 16t^2$

b) Range occurs when $y = 0$ (Ball on Ground)

$0 = 150t\sin 25 - 16t^2$

on calc intersect

$t = 3.9620462$

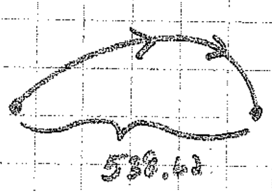
so... $X = 150(3.9620462)\cos 25$

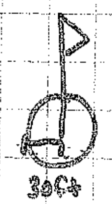
$X = 538.6249985$

200 yds = 600 ft

$600 - 538.6249985 = 61.375$ ft short of pin.

c) 20 yds = 60 ft





$$\begin{array}{r} 61.375 \\ -30 \\ \hline \boxed{31.375} \end{array}$$

ft short of the green.

The "Human Cannonball" is shot out of a cannon 10 feet above the ground with an initial velocity of 103 feet per second and at an angle of 35°.

a. What is the maximum range of the cannon?

b. How far from the launch point should a safety net be placed if the "Human Cannonball" is to land on it at a point 8 feet above the ground?

c. How long is the flight of the "Human Cannonball" from the time he is launched to the time he lands in the safety net?

a) $X = t \cdot v \cdot \cos\theta$

$\boxed{X = 103\,t\,\cos 35}$

$y = t \cdot v \cdot \sin\theta - 16t^2 + h_0$

$y = 103t\sin 35 - 16t^2 + 10$

Range is $y = 0$

$0 = 103t\sin 35 - 16t^2 + 10$

2nd Trace Int #5 $\quad t = 3.8545446$

$X = 103(3.8545446)\cos 35$

$X = 325.2181832$

$\boxed{X = 325.22\text{ ft}}$

b)

$8 = 103t\sin 35 - 16t^2 + 10$

$t = 3.7259468$

$X = 103(3.7259468)\cos 35$

$X = 314.3880446$

$X = 314.37$ ft from launch pt.

c) 3.725 secs

Dan Carpenter, kicker for the Buffalo Bills, kicks the football with an initial velocity of 29 yards per second at an angle of $68°$ to the horizontal. Suppose the kick returner catches the ball 5 seconds later.

a. How far has the ball traveled horizontally and what is the vertical height at that time?

b. Suppose the kick returner lets the ball hit the ground instead of catching it. How long is the ball in the air and how far did it travel?

a) $X = t \cdot v \cos\theta$

$X = 29t \cos 68$

$X = \cancel{87}\, t \cos 68$

$X = 29(5) \cos 68$

$X = 54.31795605$

$\boxed{X = 54.32 \text{ yds}}$

$\boxed{X = 162.9538681 \text{ ft}}$

$y = t \cdot v \cdot \sin\theta - 16t^2 + h_0$

$y = 29t \sin 68 - 16t^2 + 0$

$y = 29t \sin 68 - 16t^2$

29 yds/sec = ×3 87 ft/sec

$y = 87(5) \sin 68 - 16(5)^2$

$y = 3.3249767 37$

$\boxed{y = 3.32 \text{ ft}}$

b) Range $y = 0$

$0 = 87t \sin 68 - 16t^2$

2nd trace intersect

$t = 5.04156$

$\boxed{t = 5.04 \text{ secs}}$

$X = 87(5.04) \cos 68$

$X = 164.257449$ ft

or 54.75 yds.